# Rayleigh-Bénard Instability and the Boussinesq Approximation

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Consider a heat conducting fluid that is stationary and in a state of equilibrium. When this fluid is heated from the bottom, the fluid's tendency is to remain stationary while conducting the heat from the bottom toward the top. If one continues to heat the bottom, the temperature difference between the top and the bottom eventually reaches a critical value above which the fluid can no longer support the equilibrium state and convection begins. This phenomenon, known as the Rayleigh-Bénard instability, is the subject of this note.

## 1 The Basic Solution

We begin the mathematical study of this phenomenon by considering the governing equations of fluid dynamics in a domain $D$ defined by

$$D = \{(x, y, z) \,|\, 0 \leq z \leq d\}. \tag{1}$$

The fluid therefore is confined between two slabs (at $z = 0$ and $z = d$) and otherwise extends to infinity. The following three sets of equations are derived from the principles of conservation of mass, linear momentum and energy, respectively:

$$\left\{ \begin{aligned} \rho_t + \text{div } (\rho \mathbf{u}) &= 0, \\ \rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) &= -\nabla p + \mu \Delta \mathbf{u} - \rho g \mathbf{k}, \\ \rho c_p(\theta_t + \mathbf{u} \cdot \nabla \theta) &= \lambda \Delta \theta, \end{aligned} \right. \tag{2}$$

where $\mathbf{u} = \langle u, v, w \rangle$ denotes the velocity, $\rho$ the density, $\theta$ the temperature, $p$ the pressure, $\mu$ the kinematic viscosity, $g$ the gravitational constant, $c_p$ the specific heat, and $\lambda$ the thermal conductivity of the fluid. We have assumed the Fourier law, that heat flux $\mathbf{q} = -\lambda \nabla \theta$, in expressing the heat flux's dependence on temperature.

The term $\rho g \mathbf{k}$ takes into account the force associated with the weight of the fluid. This term, which measures the bouyancy in the flow, plays a crucial role in the stability analysis of the flow since $\rho$ changes with temperature, thus introducing a rather complex forcing term on the right-side of (2b).

Equations (2) are augmented by the boundary conditions

$$\left\{ \begin{aligned} w(x, y, 0) &= 0, \quad w(x, y, d) = 0, \\ \theta(0, x, y) &= \theta_0, \quad \theta(d, x, y) = \theta_d, \end{aligned} \right. \tag{3}$$

where $\theta_0$ is the (controlled) temperature at the bottom of the slab and $\theta_d$ is the corresponding temperature at $z = d$. Finally, we complete the mathematical formulation of the flow by assuming the initial data

$$\mathbf{u}(x,y,z,0) = \mathbf{f}(x,y,z), \quad \theta(x,y,z,0) = g(x,y,z). \tag{4}$$

The key idea in the Boussinesq approximation (see Drazin and Reid, "Hydrodynamic Stability", 1981, Cambridge University Press, pp. 32 – 52) is to assume that the density deviations are only due to temperature variations. More important, the density variations only manifest themselves in the bouyancy term in (22). We therefore assume that $\rho$ has the from

$$\rho = \rho_0(1 + \alpha(\theta_0 - \theta)), \tag{5}$$

where $\rho_0$ and $\alpha$ are nonnegative constants. So except for the term $\rho g\mathbf{k}$, we replace $\rho$ by $\rho_0$ whereever $\rho$ appears in (2). The latter equations now reduce to

$$\left\{ \begin{array}{rl} \text{div } \mathbf{u} & = 0, \\ \rho_0(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) & = -\nabla p + \mu\Delta\mathbf{u} - \rho_0(1+\alpha(\theta_0-\theta))g\mathbf{k}, \\ \theta_t + \mathbf{u}\cdot\nabla\theta & = \kappa\Delta\theta, \end{array} \right. \tag{6}$$

where $\kappa = \frac{\lambda}{c_p\rho_0}$.

Our stability analysis begins with identifying the basic solution of (6) which satisfy (3) and represents the equilibrium state of flow where the fluid is standing still and heat transfer is solely due to heat conduction. This solution $(\mathbf{u}^0, \theta^0, p^0)$ is

$$\left\{ \begin{array}{ll} \mathbf{u}^0 & \equiv \mathbf{0}, \\ \theta^0 & = \theta_0 - \frac{\theta_0-\theta_d}{d} z, \\ p^0 & = p_0 - g\rho_0(z + \frac{\alpha(\theta_0-\theta_d)}{2d} z^2), \end{array} \right. \tag{7}$$

where $p_0$ is the fluid pressure at $z = 0$.

**Problem 1**: Show that (7) satisfy the equations in (6) and the boundary conditions (3).

## 2 Linearization about the Basic Solution

Our goal in this note is to understand how the perturbations of the basic solution defined in (7) behave as solutions of the initial-boundary value problem (6), (3) and (4). To that end we consider the solution $\{\mathbf{u}, \theta, p\}$ of (6), (3) and (4) which are related to (7) through the relations

$$\left\{ \begin{array}{ll} \mathbf{u} & = \epsilon\mathbf{U}, \\ \theta & = \theta_0 - \beta z + \epsilon\Theta, \\ p & = p_0 - g\rho_0(z + \frac{\alpha\beta}{2} z^2) + \epsilon P, \end{array} \right. \tag{8}$$

where $\beta = \frac{\theta_0-\theta_d}{d}$ and $\epsilon$ is a small real number. The triple $\{\mathbf{U}, \Theta, P\}$ represent the perturbations away from the basic solution (7). Since $\{\mathbf{u}, \theta, p\}$ are solutions of the full problem they must satisfy (6). We now substitute (8) into (6) and retain only terms of order $\epsilon$ to get

$$\begin{cases} \text{div } \mathbf{U} &= 0, \\ \frac{\partial \mathbf{U}}{\partial t} &= -\frac{1}{\rho_0}\nabla P + \alpha\Theta g\mathbf{k} + \frac{\mu}{\rho_0}\Delta\mathbf{U}, \\ \frac{\partial \Theta}{\partial t} &= \beta u_3 + \kappa\Delta\Theta. \end{cases} \tag{9}$$

**Problem 2**: Show that the perturbation equations in (9) follow directly from substitution of (8) into (6).

## 3 Non-dimensionalization

Next we non-dimensionalize the dependent and independent variables in (9) according to the following set of relations:

$$\begin{cases} \bar{\mathbf{x}} &= \frac{\mathbf{x}}{d}, \\ \bar{t} &= \frac{\kappa}{d^2}t, \\ \bar{\mathbf{u}} &= \frac{d}{\kappa}\mathbf{U}, \\ \bar{\theta} &= \frac{1}{\beta d}\Theta, \\ \bar{p} &= \frac{d^2}{\rho_0\kappa^2}P. \end{cases} \tag{10}$$

**Problem 3**: Show that the barred quantities in (10) are all non-dimensional.

Application of the chain rule of differentiation transforms each equation in (9) to its non-dimensional equivalent. For instance, the first equation, $\nabla \cdot \mathbf{U} = 0$, takes the form

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{U} \\ &= \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} + \frac{\partial U_3}{\partial x_3} \\ &= \frac{1}{d}\left(\frac{\partial U_1}{\partial \bar{x}_1} + \frac{\partial U_2}{\partial \bar{x}_2} + \frac{\partial U_3}{\partial \bar{x}_3}\right) \\ &= \frac{\kappa}{d^2}\left(\frac{\partial \bar{U}_1}{\partial \bar{x}_1} + \frac{\partial \bar{U}_2}{\partial \bar{x}_2} + \frac{\partial \bar{U}_3}{\partial \bar{x}_3}\right) \\ &= \frac{\kappa}{d^2}\bar{\nabla} \cdot \bar{U}. \end{aligned}$$

Thus the first equation in (9) remains invariant under the change of coordinates (10), giving us

$$\bar{\nabla} \cdot \bar{\mathbf{U}} = 0.$$

The remiainng equations in (9) are treated similarly.

**Problem 4**: Show that the dimensional equations in (9) reduce to the following non-dimensional equations:

$$\begin{cases} \bar{\nabla} \cdot \bar{\mathbf{U}} &= 0, \\ \frac{\partial \bar{u}_1}{\partial \bar{t}} &= -\frac{\partial \bar{p}}{\partial \bar{x}_1} + \frac{\nu}{\kappa}\bar{\Delta}\bar{u}_1, \\ \frac{\partial \bar{u}_2}{\partial \bar{t}} &= -\frac{\partial \bar{p}}{\partial \bar{x}_2} + \frac{\nu}{\kappa}\bar{\Delta}\bar{u}_2, \\ \frac{\partial \bar{u}_3}{\partial \bar{t}} &= -\frac{\partial \bar{p}}{\partial \bar{x}_3} + R\frac{\nu}{\kappa}\bar{\theta} + \frac{\nu}{\kappa}\bar{\Delta}\bar{u}_1, \\ \frac{\partial \bar{\theta}}{\partial \bar{t}} &= \bar{u}_3 + \bar{\Delta}\bar{u}_3, \end{cases} \tag{11}$$

where $R$, called the **Rayleigh number**, is

$$R = \frac{g\alpha\beta d^4}{\kappa\nu}. \tag{12}$$

We also define the **Prandtl number** $\sigma$ as

$$\sigma = \frac{\nu}{\kappa} \tag{13}$$

**Problem 5**: complete the computations that lead to (11).

With the new definitions of $R$ and $\sigma$ in hand the equations (11) on the following form (we drop the bars from this point on):

$$\begin{cases} \nabla \cdot \mathbf{U} & = 0, \\ \frac{\partial \mathbf{u}}{\partial t} & = -\nabla p + R\sigma\theta\mathbf{k} + \sigma\Delta\mathbf{u}, \\ \frac{\partial \theta}{\partial t} & = u_3 + \Delta\theta. \end{cases} \tag{14}$$

The following identities will be used in reducing (14).

**Problem 6**: Let $\mathbf{u}$, $p$ and $\theta$ be smooth functions. Then the following identities hold:

1. $\nabla \times \nabla p = \mathbf{0}$.
2. $\nabla \times (\theta\mathbf{k}) = \nabla\theta \times \mathbf{k}$.
3. $\nabla \times (\nabla \times (\theta\mathbf{k})) = -\Delta\mathbf{k} + \nabla\theta_z$. In particular, the third component of $\nabla \times (\nabla \times (\theta\mathbf{k})) = -(\theta_{x_1x_1} + \theta_{x_2x_2})$.
4. $\nabla \times \nabla \times \mathbf{u} = -\Delta\mathbf{u} + \nabla(\nabla \cdot \mathbf{u})$.

Note that the last identity implies that $\nabla \times \nabla \times \mathbf{u} = -\Delta\mathbf{u}$ when the flow is incompressible. With this in mind we apply the operator $\nabla \times \nabla \times$ to the second equation in (14), apply the remaining identities in Problem 6 and get

$$\frac{\partial}{\partial t}(\Delta u_3) = R\sigma\Delta_1\theta + \sigma\Delta^2 u_3$$

where $\Delta_1 u = u_{x_1x_1} + u_{x_2x_2}$. We now rewrite the above equation slightly and group it with the last equation in (14):

$$\begin{cases} (\frac{\partial}{\partial t}\Delta - \sigma\Delta^2)u_3 & = R\sigma\Delta_1\theta, \\ (\frac{\partial}{\partial t} - \Delta)\theta = u_3. \end{cases} \tag{15}$$

We apply the operator $(\frac{\partial}{\partial t} - \delta)$ to the first equation in (15) and use the second equation in (15) to get a single equation for $u_3$:

$$(\frac{\partial}{\partial t} - \Delta)(\frac{\partial}{\partial t}\Delta - \sigma\Delta^2)u_3 = R\sigma\Delta_1 u_3. \tag{16}$$

**Problem 7**: Show that $\theta$ satisfies the same differential equation as $u_3$ does.

# 4 Separation of Variables

We now seek solutions to (16) and the last equation in (14) in the form

$$u_3 = e^{st} f(x,y) w(z), \quad \theta = e^{st} f(x,y)\theta(z) \tag{17}$$

where $w$ satisfies the rigid boundary conditions

$$w(0) = w(1) = 0. \tag{18}$$

Substituting (**??**) into the last equation in (14) leads to

$$sf\theta = fw + \theta\Delta_1 f + f\theta'',$$

which after dividng by $f\theta$ and recalling that $f$ depends only on $x$ and $y$ and $\theta$ only on $z$ we conclude that

$$\frac{\Delta_1 f}{f} = -a^2, \qquad \frac{\theta''}{\theta} + \frac{w}{\theta} - s = a^2$$

where $a^2$ is the eigenvlaue of this problem. The above equations are equivalent to

$$\Delta_1 f + a^2 f = 0, \qquad \theta'' + w - s\theta = a^2\theta. \tag{19}$$

Next we substutute (17) into (16) to get

$$s^2 f(\frac{d^2w}{dz^2} - a^2 w) - s(\sigma+1) f(\frac{d^4w}{dz^4} - 2a^2\frac{d^2w}{dz^2} + a^4 w) + \sigma f(\frac{d^6w}{dz^6} - 3a^2\frac{d^4w}{dz^4} + 3a^4\frac{d^2w}{dz^2} - a^6 w) = -R\sigma a^2 fw.$$

we divide the above equation by $\sigma f$ to get

$$\frac{d^6w}{dz^6} - (3a^2 + s(1+\frac{1}{\sigma}))\frac{d^4w}{dz^4} + (3a^4 + 2a^2 s(1+\frac{1}{\sigma}) + \frac{s^2}{\sigma})\frac{d^2w}{dz^2} + (Ra^2 - a^6 - a^4 s(1+\frac{1}{\sigma}) - \frac{a^2 s^2}{\sigma})w = 0. \tag{20}$$

Since (20) is linear with constant coefficients and $w$ must satisfy zero boundary conditions, we seek solutions of this equation of the form $w(z) = \sin n\pi z$ leaving us with the polynomial equation

$$-n^6\pi^6 - (3a^2 + s(1+\frac{1}{\sigma}))n^4\pi^4 - (3a^4 + 2a^2 s(1+\frac{1}{\sigma}) + \frac{s^2}{\sigma})n^2\pi^2 + (Ra^2 - a^6 - a^4 s(1+\frac{1}{\sigma}) - \frac{a^2 s^2}{\sigma}) = 0.$$

Additionally, we note that the solution (17) will be marginally stable when $s = 0$ so we will $s = 0$ in the above equation to determine a relationship between $R$ and $a$ that would lead to this limiting case. We get

$$-n^6\pi^6 - 3a^2 n^4\pi^4 - 3a^4 n^2\pi^2 + Ra^2 - a^6,$$

from which we obtain the relation

$$R(a) = 1\frac{1}{a^2}(a^2 + n^2\pi^2)^3. \tag{21}$$

The function $R$ in (21) attains its minimum when $a$ has the critical value

$$a = \frac{n\pi}{\sqrt{2}} \tag{22}$$

Substituting (22) in (21) leads to the critical Rayleigh number

$$R_c = \frac{27n^4\pi^4}{4} \tag{23}$$

at which marginal instability of the steady solution (7) occurs. When $n = 1$ the critical Rayleigh number is $\frac{27\pi^4}{4}$ which occurs at $a = \frac{\pi}{\sqrt{2}}$. We have proved the following theorem.

**Theorem**: The basic solution (7) is linearly unstable as the solution of the fully nonlinear system (6) if the Rayleigh number $R$, which is $\frac{g\alpha\beta d^4}{\kappa\nu}$, is greater than or equal to $\frac{27\pi^4}{4}$.